Integration-by-parts (IBP) reduction is a crucial step for the loop amplitude computation.

Recalled that we defined that

$$G[n_1, \ldots, n_k] = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{n_1} \ldots D_k^{n_k}} \tag{1}$$

For arbitrary $n_i \in \mathbb{Z}$'s, $i = 1, \ldots k$, we call the set of all such $G[n_1, \ldots, n_k]$ an integral family.

Suppose the we have used an integrand reduction process, like the OPP-reduction or Passarino-Veltman reduction, to reduce the tensor production to a Feynman integral with external momenta only (without polarization vectors or the auxiliary ω vector). Thus the integral only contains external momenta. Furthermore, define E as the linearly independent vectors. To make the IBP reduction work we have to add some irreducible scalar products (ISPs), to set

$$k = LE + \frac{L(L+1)}{2}\,. \tag{2}$$

To goal of IBP reduction is to linearly reduce integrals in a family to a finite number of "simple integrals". Note the for the one-loop case, after the OPP reduction, all n_i's are either 0 or 1, so there is no much IBP work to do. However, for the two-loop case, even with the OPP plus the Groebner basis reduction, the output contains a large number of integrals and should be reduced by IBPs.

Later we will see that for differential equations of Feynman integrals, there would be a lot of Feynman integrals with $n_i > 1$. Usually these integrals should be reduced by IBPs also.

In practice, usually the IBP reduction is most time and RAM consuming part for an amplitude computation.

I. FUNDAMENTAL IBPS

For an integral family, the fundamental IBPs are

$$0 = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v^\mu}{D_1^{n_1} \ldots D_k^{n_k}} \tag{3}$$

Here $i = 1, \ldots L$ and the vector v's can be chosen in $\{p_1, \ldots p_E, l_1 \ldots l_L\}$. Hence there are $L(L+E)$ fundamental IBPs. Note that for one fundamental IBP actually contains infinite numbers of IBP relations, since n_i's are arbitrary integers.

The derivatives in (3) should be rewritten as a linear of D_i's. This involves some simple computations.

II. SECTORS

As a linear system, the IBPs have the structure of a big upper block-triangular matrix. To see this, we introduce the concept of sectors of integrals.

For an integral $G[n_1, \dots n_k]$, we define its sector as $(s_1, \dots s_k)$. Here

$$s_i = \begin{cases} 1 & n_i > 0 \\ 0 & n_i \leq 0 \end{cases} \tag{4}$$

The concept of sector is equivalent to the topology of Feynman diagrams.

If for two *distinct* sectors $S^{(1)} = (s_1^{(1)}, \dots, s_k^{(1)})$ and $S^{(2)} = (s_1^{(2)}, \dots, s_k^{(2)})$,

$$s_i^{(1)} > s_i^{(2)}, \quad 1 \leq i \leq n \tag{5}$$

The we call $S^{(1)}$ a super-sector of $S^{(2)}$, and $S^{(2)}$ a sub-sector of $S^{(1)}$.

Usually we think that integrals in the super sectors are more "complicated". Hence if $S^{(2)}$ a sub-sector of $S^{(1)}$, we define

$$I_1 \succ I_2, \quad \forall I_1 \in S^{(1)}, \quad I_2 \in S^{(2)} \tag{6}$$

However, the super-sub-sector ordering for the sectors is not a total ordering. We need to by hand extend it to a total ordering of all the sectors.

For two integrals in the same sector, we also define a ordering to formally indicate the complication. We call the unique integral I in a sector with n_i's either zero or one, the "corner integral. Then we define the corner integral as the simplest integral in the sector. Then we give an ordering for the other integrals, by the distance from the corner integral.

By the procedure above, we have a total ordering of all integrals in one family. The total ordering is not unique. However, we emphasize that, *traditionally* there are two crucial features which are shared in all integral total orderings:

1. The super-sub-sector relation is respected by an integral ordering.

2. The corner integral is the lowest integral for a given sector.

We have the total ordering of all integrals in a family, and the IBP relations between integrals in a family. The linear coefficients must be rational functions of Mandelstam variables, masses and the spacetime dimension. As usual, we list the IBP relation coefficients in a matrix, where the row is for the relation and the column is the for the integrals *sorted in the total integral ordering, from higher to lower.*

Therefore the IBP coefficient matrix is block upper-triangular.

III. IBP REDUCTION AND MASTER INTEGRALS

The goal of IBP reduction is to express “complicated” Feynman integrals to a linear combination of “simple” Feynman integrals. Given enough number of IBP relations, we run the Gaussian elimination of the IBP coefficient matrix, to get the IBP reduction result.

Since the matrix is block upper-triangular, we carry out the Gaussian elimination block by block. Furthermore, we may also carry out the Gaussian elimination in parallel for several sectors without the “super-sub” sector relation.

Since the coefficients contain the Mandelstam variables, masses and the spacetime dimension, the Gaussian elimination is a very heavy computation.

If an integral cannot be expressed as a linear combinations of lower integrals, i.e., the corresponding matrix column does not have a pivot after Gaussian elimination, then we can this integral a “master integral”. Note that the master integral definition depends on the ordering choice of the integral ordering.

Smirnov’s proved that for a given integral family the number of master integral is always a finite number. The different integral orderings do not change this number.

Roman Lee claimed that the number of master integrals in one sector equals the number of

$$\{\text{critical points of } F\} \equiv \{\frac{\partial F}{\partial z_i} = 0 \text{ and } \ F \neq 0\} \tag{7}$$

where F is the Lee-Pomeransky polynomial or the Baikov polynomial on cut. However, this claim has counter examples.

IV. ZERO SECTORS

We treat any scaleless integral in dimensional regularization as zero. This statement is to be understood as an analytic continuation.

For instance, the massless tadpole, the no-propagator scale Feynman integral are all zero in dimensional regularization. Furthermore, if all integrals in one sector are zero, we call this sector “zero sector”.

From the partial fraction, we see that if the corner integral in a sector is zero, then actually all integrals in this sector is zero and thus this sector is a zero sector.

It is important to identify zero sectors in an integral family, so that we have the IBP reduction boundary.

V. SYMMETRIES

Frequently, we have symmetries between different Feynman integrals. The symmetry may come from the graph symmetry or the symmetry of symanzik polynomials.

For example, consider the two-loop massless double box diagram with the propagators,

$$\begin{aligned} D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2 \quad D_3 = (l_1 - p_1)^2 \quad D_4 = (l_1 - k_1 - k_2)^2, \\ D_5 = (l_2 + k_1 + k_2)^2, \quad D_6 = (l_2 - k_4)^2, \quad D_6 = l_2^2 \quad , D_7 = (l_1 + l_2)^2 \end{aligned} \tag{8}$$

as well as two ISPs

$$D_8 = (l_1 + k_4)^2, \quad D_9 = (l_2 + k_1)^2 \,. \tag{9}$$

This diagram has the left-right and up-down symmetries. For the left-right symmetry, naively we see that

$$k_1 \to k_4, \quad k_2 \to k_3, \quad k_3 \to k_2, \quad k_4 \to k_1, \tag{10}$$

and

$$D_1 \to D_6, \quad D_2 \to D_5, \quad D_3 \to D_4, \quad D_4 \to D_3, \quad D_5 \to D_2 \quad D_6 \to D_1 \quad D_7 \to D_7 \tag{11}$$

as well as the transformation of ISPs. We see that it induces the transformation,

$$l_1 \to l_2, \quad l_2 \to l_1 \tag{12}$$

It implies that,

$$\begin{aligned} & G[n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9](p_1, p_2, p_3, p_4) \\ & = G[n_6, n_5, n_4, n_3, n_2, n_1, n_7, n_9, n_8](p_4, p_3, p_2, p_1) \end{aligned} \tag{13}$$

However, we know that the Feynman integral only depends on Lorentz invariants. It is clear that $p_4 \cdot p_3 = p_1 \cdot p_2$ and $p_4 \cdot p_2 = p_1 \cdot p_3$. Therefore

$$G[n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9] = G[n_6, n_5, n_4, n_3, n_2, n_1, n_7, n_9, n_8] \tag{14}$$

This relation provides the symmetry relation in the double box sector, like,

$$G[1, 1, 1, 1, 1, 1, 1, 0, -1] = G[1, 1, 1, 1, 1, 1, 1, -1, 0] \,. \tag{15}$$

What is less obvious is the symmetry relating different sectors. From (14), we see that

$$G[1,1,0,1,1,0,1,0,0] = G[0,1,1,0,1,1,1,0,0] \tag{16}$$

$$G[1,1,1,0,1,0,1,0,0] = G[0,1,0,1,1,1,1,0,0] \tag{17}$$

That means the sector $(1,1,0,1,1,0,1,0,0)$ is equivalent to $(0,1,1,0,1,1,1,0,0)$.

Sometime the symmetry induces complicated relations between integrals, if the symmetry group is large. For instance, consider the 2-loop uniformly massive subset diagram with the propagators

$$D_1 = l_1^2 - m^2, \quad D_2 = l_2^2 - m^2, \quad D_3 = (l_1 + l_2 + p)^2 - m^2, \tag{18}$$

as well as the IPS,

$$D_4 = (l_1 + p)^2, \quad D_5 = (l_2 + p)^2, \tag{19}$$

with $p^2 = s$. Consider the symmetry,

$$l_1 \to -l_1 - l_2 - p, \quad l_2 \to l_2 \tag{20}$$

This transformation does not change the integral measure. Under this symmetry,

$$\begin{gathered} D_1 \to D_3, \quad D_2 \to D_2, \quad D_3 \to D_1 \\ D_4 \to (l_1 + l_2)^2 = D_1 + D_2 + D_3 - D_4 - D_5 + 3m^2 + s, \quad D_5 \to D_5 \end{gathered} \tag{21}$$

Therefore we have complicated symmetry relations like

$$\begin{aligned} G[1,1,1,-1,0] &= G[0,1,1,0,0] + G[1,0,1,0,0] + G[1,1,0,0,0] \\ &-G[1,1,1,-1,0] - G[1,1,1,0,-1] + (3m^2 - s)G[1,1,1,0,0] \end{aligned} \tag{22}$$

In practice, we can use polynomial mapping algorithms to identity equivalent Symanzik polynomial, in order to find symmetries.